WHAT IS CHAOS?
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INTRODUCTION
Can a butterfly flapping her wings in Beijing today bring storms in New York next month? Traditionally, when physicists saw complex results, they searched for complex causes. In the last 20 years, scientists have created an alternative set of ideas that simple systems give rise to a complex behavior, and a complex systems give rise to simple behavior and an element of disorder within order was introduced. Chaos refers to the condition of a system in extreme confusion and disorder. It is the formless and disordered state of matter before the creation of the universe [1].

Chaos is the term used to describe the apparently complex behavior of what we consider to be simple and well-behaved systems or for the circumstance that the motion of very simple dynamical systems can not always be predicted far into the future [2]. In other words we can say that chaotic is a term assigned to that class of motions in deterministic physical and mathematical systems whose time history has a sensitive dependence on initial conditions. The sensitive dependence on initial conditions of chaotic systems is more popularly known as the butterfly effect. This phenomenon was first discovered by Edward Lorenz during his investigation into a system of coupled ordinary differential equations used as a simplified model of 2D thermal convection, known as Rayleigh-Benard convection. These equations are now called the Lorenz equations, or Lorenz model.

The Duffing Oscillator
This nonlinear oscillator is an example of a system that becomes chaotic when driven by a periodic force. In fig.(1a), a sliding block oscillator model is shown. It consists of mass $m$, attached to a support (for example a wall) by a spring having stiffness $s$, a frictional resistance $r$ between the block and the sliding surface and a force, acting on the mass. Such model is capable of producing chaos under appropriate conditions. Initially the frictional resistance and forcing is set equal to zero. When the mass is moved from its rest position, at the same time a restoring force is offered by the stretched spring, which wants to pull the mass back to its rest position. The restoring force is equal to the spring constant $s$ multiplied by the extension $x$ of the spring. Mathematically: $F_s = -sx$

The negative sign in the above equation indicates that the restoring force and the extension are both opposite in direction. If one pulls the mass to some new position, for example a distance $A$, from the stable equilibrium position $x$ (rest position) and then release it. The mass will oscillate to and fro between $+A$ and $-A$ from its rest position. The time series of such kind of undamped simple harmonic motion is given in the fig.(1b). Now on including the fractional resistance of the block sliding on the surface by the addition of a fractional or damping component $r$ and then releasing the mass from position $A$. This time, although the mass will oscillate to and fro, but with ever decreasing amplitude. The fractional resistance force is assumed to be proportional to the velocity of the block. Due to friction the amplitude of oscillation decays exponentially. The time series of this damped simple harmonic motion is shown in fig.(1c).

The oscillation may be kept going on indefinitely by applying a time varying force, e.g a sinusoidal force, to the model system. Now if one initiates the oscillation from an arbitrary initial displacement, then the time development generally consists of two regions. The first one is called an Initial Transient Oscillation, which takes the dynamical system from the initial condition to the second, its Post Transient Oscillation, as shown in the fig.(1d).

![Fig. 1 Sliding Block Oscillator Model. (a) The forced, damped oscillator model. (b) The time series of simple harmonic oscillations without damping. (c) The time series of lightly damped simple harmonic motion. (d) Typical transient and post transient solutions. [3].](image)
From its initial condition, the dynamical system is attracted towards this post transient solution or attractor. Post transient attractors are distinctive elements of dissipative dynamical systems and in the system under consideration, the dissipation occurs because of the friction term in the equation of motion. Once on the post transient solution, the system stays there unless disturbed by an external force.

![Fig. 2](image1)

**Fig. 2** The time series for the Duffing Oscillator. [3].

In this case, we have a periodic post transient solution. A chaotic post transient solution can be obtained by the addition of one further ingredient to the dynamical system under consideration, which is the non-linearity. Hence considering a nonlinear spring, where the spring restoring force is proportional to the cube of the displacement. Such kind of simple forced, damped nonlinear oscillator is called the Duffing Oscillator [3]. The equation of motion of the Duffing oscillator is given as:

$$m\left(\frac{d^2x}{dt^2}\right)+r\left(\frac{dx}{dt}\right)+sx^3=Af \cos \omega t$$

The dynamical system, which is described by the above equation, may exhibit either periodic or non-periodic (chaotic) motion depending upon the values of the parameter, selected for the system. Simplification of the equation of motion of Duffing oscillator is obtained by setting the mass $m$, stiffness $s$ of the spring and the angular frequency $\omega$ equal to unity, then the above equation becomes:

$$\left(\frac{d^2x}{dt^2}\right)+r\left(\frac{dx}{dt}\right)+x^3=Af \cos \omega t$$

In this case, we have only two control parameters, the damping co-efficient and the forcing amplitude $A$. The regions of periodic and chaotic oscillations can be obtained by changing the above two control parameters. As an example, for parameter values of $r = 0.08$ and $A = 0.20$ the behavior of the system is periodic. While for $r=0.05$ and $A=7.50$, chaotic behavior is found. More than one periodic, post transient solution is oftenly possible in such a system, where each one representing the long term motion of the oscillator. The post transient solution, picked up by the oscillator, depends upon the initial conditions of the system under consideration, i.e. the initial displacement and velocity. Basin of attraction of the attractor is the set of initial conditions, which leads to a particular post transient solution. In case of the Duffing oscillator, as given by the above equation with control parameters fixed at $r=0.08$ and $A=0.20$, it would be attracted towards many post transient states (two of the transient and post transient states are given in the fig.2).

The oscillation settles to the chaotic region after the disappearance of the high frequency transient. The chaotic oscillations are irregular and exotic (unpredictable). Phase portraits of the Duffing oscillator are also shown in the fig.(3). The dynamical system evolves through time as a continuous curve in the phase plane.

![Fig. 3](image2)

**Fig. 3** Phase portraits of the Duffing oscillator. [3].

In the fig.(3), the initial conditions are given as large dots. Limit cycle attractors are given by post transient behavior of the periodic cases as closed loops in the phase plane. Hence, the closed loops, corresponding to the post transient solution, are also shown.
transient behavior of the system are a portrait of the periodic attractor. The phase portrait of the attractor for the chaotic case is also given in the fig.(3). It is known as strange or chaotic attractor and it shows all the possible final states of the chaotic solution. The chaotic oscillations of the Duffing oscillators although seems to be unpredictable, but they are not truly unpredictable. Whenever, we begin a chaotic solution on a particular set of initial conditions, we get identical time series. The system is said to be deterministic because we can accurately determine the long-term behavior of any chaotic system when we know the initial conditions exactly.

![Fig. 4](image)

**Fig. 4** Sensitive dependence on initial conditions. [3].

Any slight change in the initial conditions can quickly lead to a completely different time series in a certain chaotic system, a phenomenon known as sensitive dependence on initial conditions and is typical for any chaotic motion. In fig.(4), the two solutions initially appear to follow an identical path. But as time passes, the solution paths start to depart, giving rise to completely different long-term behavior. Any tiny error in the measurement of the initial conditions of a real dynamical system leads rapidly to a lack of predictability of its long-term behavior. Since we cannot measure any real dynamical system with infinite precision, the long-term prediction of chaotic motion in such systems is impossible, even if we know their equations of motion exactly.

**Lorenz Model**

Certain hydrodynamical systems exhibit steady-state flow patterns, while others oscillate in a regular periodic fashion. Still others vary in an irregular, seemingly haphazard manner, and even when observed for long period of time do not appear to repeat their previous history. These modes of behavior may all be observed in the familiar rotating-basin experiments, described by Fultz et al. (1959) and Hide (1958). In these experiments, a cylindrical vessel containing water is rotated about its axis, and is heated near its rim and cooled near its center in a steady symmetric fashion. Under certain conditions the resulting flow is as symmetric and steady as the heating which gives rise to it. Under other conditions a system of regularly spaced waves develops, and progresses at a uniform speed without changing its shape. Under still different conditions an irregular flow pattern forms, moves and changes its shape in an irregular non-periodic manner [3]. In 1960 the same phenomenon was observed by Edward Lorenz during his investigation into a system of coupled ordinary differential equations used as a simplified model of 2D thermal convection, known as Rayleigh-Benard convection [4]. These equations are now called the Lorenz equations, or Lorenz Model.

Before describing Lorenz model, first we concentrate on fluid system, where chaos has shed light upon the transition from the ordered to a highly disordered state. A fluid is a non-rigid, continuous interconnected mass, which is generally may exhibit either ordered (laminar) or disordered (turbulent) flow. Laminar flows are characteristic of slow-moving or highly viscous flows where the fluid particles move in an ordered fashion, sliding over themselves in sheets. As an example of a laminar flow, think of a slow-moving viscous flow from a spilled honey pot on a table with a slight incline. Turbulent flows, on the other hand, are characteristic of fast-moving or low-viscosity flows, where small disturbances in the flow will blow up causing the fluid particles to move in an unpredictable fashion, mixing themselves up from one point in the flow to the next. As an example of a turbulent flow, think of the highly energetic flow in the plunge pool at the base of a waterfall. Fluid flow is governed by a set of nonlinear partial differential equations, known as the Navier-Stokes equations.

As stated above, flowing fluids exist in either of two states: a laminar state, where the fluid flows in an orderly and predictable fashion; or a turbulent state, where the fluid particles move in a disorderly fashion and rapid decorrelation is evident within the flow field in both space and time. Much attention has focused on the possibility of explaining the transition from the laminar state to turbulence in terms of the transition from regular to chaotic motion [2].
From different experiments it was found that the key dynamical control parameter for a fluid system is the Reynolds number $R$. This is the ratio of inertial to viscous forces in the flow and is defined as:

$$R = \frac{\rho V D}{\mu}$$

where $\rho$ is the density of the fluid, $V$ is a characteristic flow velocity, $D$ is a characteristic length scale of the problem and $\mu$ is the dynamic velocity of the fluid. Below a critical value of the Reynolds number, the viscous forces dominate and the flow is laminar. Above the Reynolds number, inertial forces dominate and the flow is turbulent. As the Reynolds number is increased, a transition from laminar to turbulent flow takes place. This transition may appear either suddenly or gradually depending upon the geometry and the presence of noise in the system. With careful variation of the Reynolds number, chaos may be observed at the transition between laminar and turbulent flow in some fluid system.

In Lorenz model, a Rayleigh-Benard convection between two horizontal plates is considered. The bottom plate is at a higher temperature than that of the top plate. For small differences between the two temperatures, heat is conducted through the stationary fluid between the plates. However, when the temperature difference becomes large enough, buoyancy forces within the heated fluid overcome internal fluid viscosity and a pattern of counter-rotating, steady re-circulating vortices is set up between the plates. Lorenz noticed that, in his simplified mathematical model of Rayleigh-Benard convection, very small variations in the initial conditions blew up and quickly led to enormous differences in the final behavior. He reasoned that if this type of behavior could occur in such a simple dynamic system, then it may also be possible in a much more complex physical system involving convection such as in the weather system. Thus a very small perturbation, caused for instance by a butterfly flapping its wings, would lead rapidly to a complete change in future weather patterns. The Lorenz equations are:

$$\begin{align*}
\frac{dx}{dt} &= \alpha(y-x) \\
\frac{dy}{dt} &= (r-z)x - y \\
\frac{dz}{dt} &= xy - bz
\end{align*}$$

This system has two non-linearities, the $xz$ term and the $xy$ term, and exhibits both periodic and chaotic motion depending upon the values of the control parameters: $\alpha$ is the Prandtl number which relates the energy losses within the fluid due to viscosity to those due to thermal conduction, $r$ corresponds to the dimensionless measure of the temperature difference between the plates known as the Rayleigh number, and $b$ is related to the ratio of the vertical height of the fluid layer to the horizontal extent of the convective rolls within it. Note also that the variables $x$, $y$ and $z$ are not spatial co-ordinates but rather represent the convective overturning, horizontal temperature variation, and vertical temperature variation respectively.

Lorenz experimented that below a critical value of the control parameter $r$, the system decays to a steady, non-oscillating state. Once $r$ increases beyond its critical value, continues oscillatory behavior occurs. A value of $r=28$ produces aperiodic behavior which Lorenz called deterministic non-periodic flow and which is referred to as chaos. From the chaotic system phase portrait, we observe that the trajectories of the chaotic solution lie on a strange attractor. We observe that the trajectories wind around two distinct lobes of the attractor. After a number of revolutions around one lobe of the attractor the trajectory then switches to the other lobe. It then spirals around this lobe before switching back to the first. The number of revolutions that the trajectory will make around each lobe, before returning to the other, is unpredictable.

The structure of the Lorenz strange attractor is extremely complex. The attractor seems to consist of two surfaces and these appear to merge. However, the solution trajectory can never cross itself, as this would lead to a recurrence of a former state of the dynamical system and would therefore imply a periodic cycle. To achieve non-crossing trajectories, we require a minimum of three dimensions for the phase space, so that the trajectories can avoid themselves [3].

Chaos exists in systems all around us. Even the simplest system can be subject to chaos, denying us accurate predictions of its behavior, and sometimes giving rise to astonishing structures of large-scale order. The most startling finding to come out of this new scientific theory is that order exists within chaos.

### Applications of chaos

There are many applications of chaos, for example it is applied to increase the power of lasers, to synchronize the output of the electronic circuits. It is used to control the oscillations in chemical reactions, to stabilize the erratic beat of unhealthy animal hearts and to encode the electronic messages for secure communications.

Real life examples of chaos include weather, irregular heart-beats (controlling heart attacks mean controlling chaotic systems with small
perturbations) and brain waves. Some aspects of chaos theory come into play in developing computer models for weather prediction. Chaos is observed in various branches of astronomy. Order and chaos appear in the sun and other stars, the solar system and galaxies, up to the whole Universe. The various types of orbits in galaxies give rise to chaotic orbits. Then Chaos is discussed in various types of dissipative systems, like gas in a galaxy, chaos in relativity and cosmology, and chaos in stellar pulsations and in the solar activity. Chaos theory is applied in many scientific fields other than Physics such as Philosophy, Mathematics, Engineering, Computer-Science, Biology, Economics, Politics, Fluid Mechanics, Population Dynamics, Psychology, Robotics etc. Chaos theory has already been applied in many scientific disciplines: mathematics, biology, computer science, economics, engineering, philosophy, physics, politics, population dynamics, psychology and robotics etc.

For the first time physicists have shown that well structured chaos can be initiated in a photonic integrated circuit. The output of a semiconductor laser is normally regular. However, if certain laser parameters are tweaked, such as by modulating the electric current pumping the laser or by feeding back some of the laser's light from an external mirror, the overall laser output will become chaotic; that is, the laser output will be unpredictable. To make the chaos even more dramatic and exploitable, scientists at the Technische Universiteit Eindhoven (in the Netherlands) use paired lasers, lasers built very close to each other on a chip in such a way that each affects the operation of the other. The Eindhoven chip, using the paired-laser mutual-perturbation approach to triggering chaos, is the first to exhibit chaos. Looking ahead to the day when opto-photonic chips are covered with thousands or millions of lasers, the Eindhoven approach could allow troubleshooters to pinpoint the whereabouts of misbehaving lasers, not only that but possibly even exploit localized chaotic effects to their advantage. Other possible uses for chip-based chaos will be the business of encryption, tomography, and possibly even in the establishment of multi-tiered logic protocols, those based not on just on the binary logic of 1s and 0s but on the many intensity levels corresponding to the broadband output of the chaotic laser system. Controlling a system becomes possible when one acquires enough information about the system, and then applies the information, to keep uncertainties in the system’s properties at manageable levels. For every single bit of information, there is a corresponding reduction in uncertainty.

Scientists at the Max Planck Institute of Quantum Optics, investigating the chaotic behaviour of the quantum world, have been able to give the first ever demonstration of quantum chaos during atom ionization. Using laser light, they released electrons from rubidium in a strong electromagnetic field. The researchers measured typical fluctuations in the electron current, which are subject to the frequency of the laser light, and which arose from the chaotic movement of the electrons. The experiment is based on an experiment from the early days of quantum mechanics demonstrating the photoelectric effect.

REFERENCES
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